1. A chicken lays n eggs. Each egg independently does or doesn’t hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn’t survive (independently of the other eggs), with probability s of survival. Let N ⇠ Bin(n, p) be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don’t survive (so X + Y = N). Find the marginal PMF of X, and the joint PMF of X and Y . Are they independent?

Answer :

we need to determine the marginal PMF of XXX and the joint PMF of XXX and YYY, and then check if XXX and YYY are independent.

**Definitions and Given Information**

* NNN: Number of eggs that hatch.
* XXX: Number of chicks that survive.
* YYY: Number of chicks that hatch but do not survive.
* N∼Bin(n,p)N \sim \text{Bin}(n, p)N∼Bin(n,p) (number of eggs that hatch)
* X+Y=NX + Y = NX+Y=N
* Probability of hatching = ppp
* Probability of survival given hatching = sss

**Marginal PMF of XXX**

To find the marginal PMF of XXX, we need the probability mass function (PMF) of XXX alone.

1. **Determine the distribution of XXX:**

Given that NNN follows a binomial distribution and that each chick that hatches survives with probability sss, the number of surviving chicks XXX given N=nN = nN=n follows a binomial distribution:

X∣N=n∼Bin(N,s)X | N = n \sim \text{Bin}(N, s)X∣N=n∼Bin(N,s)

where N∼Bin(n,p)N \sim \text{Bin}(n, p)N∼Bin(n,p).

1. **Calculate the marginal PMF of XXX:**

First, find the PMF of XXX by summing over all possible values of NNN:

P(X=x)=∑n=xnP(X=x∣N=n)⋅P(N=n)P(X = x) = \sum\_{n=x}^{n} P(X = x | N = n) \cdot P(N = n)P(X=x)=n=x∑n​P(X=x∣N=n)⋅P(N=n)

Here:

P(X=x∣N=n)=(nx)sx(1−s)n−xP(X = x | N = n) = \binom{n}{x} s^x (1 - s)^{n - x}P(X=x∣N=n)=(xn​)sx(1−s)n−x

and

P(N=n)=(nn)pn(1−p)n−nP(N = n) = \binom{n}{n} p^n (1 - p)^{n - n}P(N=n)=(nn​)pn(1−p)n−n

where N∼Bin(n,p)N \sim \text{Bin}(n, p)N∼Bin(n,p).

Thus:

P(X=x)=∑n=xn(nx)sx(1−s)n−x(nn)pn(1−p)n−nP(X = x) = \sum\_{n=x}^{n} \binom{n}{x} s^x (1 - s)^{n - x} \binom{n}{n} p^n (1 - p)^{n - n}P(X=x)=n=x∑n​(xn​)sx(1−s)n−x(nn​)pn(1−p)n−n

**Joint PMF of XXX and YYY**

To find the joint PMF of XXX and YYY, use the fact that X+Y=NX + Y = NX+Y=N and Y=N−XY = N - XY=N−X:

P(X=x,Y=y)=P(X=x,N=x+y)=P(X=x∣N=x+y)⋅P(N=x+y)P(X = x, Y = y) = P(X = x, N = x + y) = P(X = x | N = x + y) \cdot P(N = x + y)P(X=x,Y=y)=P(X=x,N=x+y)=P(X=x∣N=x+y)⋅P(N=x+y) P(X=x∣N=x+y)=(x+yx)sx(1−s)yP(X = x | N = x + y) = \binom{x + y}{x} s^x (1 - s)^yP(X=x∣N=x+y)=(xx+y​)sx(1−s)y P(N=x+y)=(nx+y)px+y(1−p)n−(x+y)P(N = x + y) = \binom{n}{x + y} p^{x + y} (1 - p)^{n - (x + y)}P(N=x+y)=(x+yn​)px+y(1−p)n−(x+y)

Thus:

P(X=x,Y=y)=(x+yx)sx(1−s)y⋅(nx+y)px+y(1−p)n−(x+y)P(X = x, Y = y) = \binom{x + y}{x} s^x (1 - s)^y \cdot \binom{n}{x + y} p^{x + y} (1 - p)^{n - (x + y)}P(X=x,Y=y)=(xx+y​)sx(1−s)y⋅(x+yn​)px+y(1−p)n−(x+y)

**Independence of XXX and YYY**

**Two random variables XXX and YYY are independent if:**

P(X=x,Y=y)=P(X=x)⋅P(Y=y)P(X = x, Y = y) = P(X = x) \cdot P(Y = y)P(X=x,Y=y)=P(X=x)⋅P(Y=y)

**To check independence:**

1. **Calculate P(Y=y)P(Y = y)P(Y=y):**

To find P(Y=y)P(Y = y)P(Y=y), sum over all possible values of XXX:

P(Y=y)=∑x=0n−yP(X=x,Y=y)P(Y = y) = \sum\_{x=0}^{n - y} P(X = x, Y = y)P(Y=y)=x=0∑n−y​P(X=x,Y=y) P(Y=y)=∑x=0n−y(x+yx)sx(1−s)y⋅(nx+y)px+y(1−p)n−(x+y)P(Y = y) = \sum\_{x=0}^{n - y} \binom{x + y}{x} s^x (1 - s)^y \cdot \binom{n}{x + y} p^{x + y} (1 - p)^{n - (x + y)}P(Y=y)=x=0∑n−y​(xx+y​)sx(1−s)y⋅(x+yn​)px+y(1−p)n−(x+y)

1. **Compare P(X=x,Y=y)P(X = x, Y = y)P(X=x,Y=y) with P(X=x)⋅P(Y=y)P(X = x) \cdot P(Y = y)P(X=x)⋅P(Y=y):**

Given the above calculations, it is generally not straightforward to show independence because the marginal probabilities and the joint probability depend on the same underlying random variable NNN.

**Conclusion:** **XXX and YYY are not independent** because XXX and YYY are both dependent on the number of eggs that hatch NNN. The joint PMF includes NNN explicitly, making XXX and YYY dependent on each other through NNN.